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NONLINEAR THEORY OF WEAKLY PERTURBED SPACE FLOW WITH AN ARBITRARY NUMBER OF NONEQUILIBRIUM PROCESSES

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Nonequilibrium space flows with an arbitrary number of relaxation processes inherent to homogeneous and heterogeneous media are considered. Sufficient conditions for the existence of flow field regions in which conventional linear theory is inapplicable, are formulated. A method is proposed for the construction for these regions of a nonlinear theory of weakly perturbed flows.

Applications of the conventional linear theory for investigating equilibrium and nonequilibrium flows is limited. This is partly due to incorrect formulation of such theory, and partly to the impossibility to define some of the flow regions by linear equations [1–3]. Modification of the conventional linear theory of equilibrium and nonequilibrium flows (see, e. g. [4]) made it possible to widen the range of its applicability and improve its accuracy. Nonlinear equations were used in [5–8] for defining nonequilibrium weakly perturbed flows with a single relaxation process. Weakly perturbed flows with an arbitrary number of nonequilibrium processes are analyzed in terms of the linear theory in [9–13].

1. Let us consider a nonstationary space flow of an inviscid and nonheat-conducting gas in which various nonequilibrium processes may take place. Let ρ be the density, p the pressure, x, y, z a system of Cartesian coordinates, t the time, $\mathbf{V}$ the velocity vector with projections u, v, w and c the speed of sound (unless otherwise stated c is assumed to be the propagation rate of small perturbations, when all relaxation processes are frozen). The fundamental equations of conservation may be presented in the following form:

$$\frac{d\rho}{dt} + \rho\nabla\mathbf{V} = L_1, \quad \rho\frac{d\mathbf{V}}{dt} + \nabla p = \mathbf{l}, \quad \frac{dp}{dt} - c^2\frac{d\rho}{dt} = L_5 \qquad (1.1)$$

where $\mathbf{l} = \{L_2, L_3, L_4\}$, and vector $\mathbf{L} = \{L_k\}$ $(k = 1, 2, \ldots, 5)$ is determined by the nonequilibrium processes. It is assumed that $\mathbf{L}$ does not contain derivatives of ρ, p and $\mathbf{V}$.

We introduce new independent variables

$$\xi_j = \xi_j(t, x, y, z) \qquad (j = 1, 2, 3, 4) \qquad (1.2)$$

with a nonzero Jacobian of transformation, and use the notation

$$A_{j1} = \frac{\partial\xi_j}{\partial t}, \quad A_{j2} = \frac{\partial\xi_j}{\partial x}, \quad A_{j3} = \frac{\partial\xi_j}{\partial y}, \quad A_{j4} = \frac{\partial\xi_j}{\partial z}$$

$$U_j = A_{j1} + \mathbf{A}_j\mathbf{V}, \quad \mathbf{A}_j = \{A_{j2}, A_{j3}, A_{j4}\}, \quad \omega_j^2 = |\mathbf{A}_j|^2$$

The system of Eqs. (1.1) in new variables assumes the form

$$\sum_{j=1}^{4}\left(\rho \mathbf{A}_j \frac{\partial \mathbf{V}}{\partial \xi_j} + U_j \frac{\partial \rho}{\partial \xi_j}\right) = L_1, \quad \sum_{j=1}^{4}\left(\rho U_j \frac{\partial \mathbf{V}}{\partial \xi_j} + \mathbf{A}_j \frac{\partial p}{\partial \xi_j}\right) = \mathbf{l} \qquad (1.3)$$

$$\sum_{j=1}^{4} U_j\left(\frac{\partial p}{\partial \xi_j} - c^2 \frac{\partial \rho}{\partial \xi_j}\right) = L_5$$

The system of Eqs. (1.3) is generally quasi-linear. The matrix of coefficients at derivatives contains five rows and 20 columns. It can be divided into four square matrices 5×5, whose elements are the same as the coefficients at derivatives of u, v, w, p and ρ with respect to ξ_i

$$\begin{Vmatrix} \rho A_{i2} & \rho A_{i3} & \rho A_{i4} & 0 & U_i \\ \rho U_i & 0 & 0 & A_{i2} & 0 \\ 0 & \rho U_i & 0 & A_{i3} & 0 \\ 0 & 0 & \rho U_i & A_{i4} & 0 \\ 0 & 0 & 0 & U_i & -c^2 U_i \end{Vmatrix} \qquad (1.4)$$

Let us calculate the determinant of this matrix

$$\Delta_i = \rho^3 U_i \left(U_i^2 - c^2 \omega^2\right)$$

The system of Eqs. (1.3) can be solved for derivatives with respect to ξ_i, if $\Delta_i \neq 0$. When $\Delta_i = 0$, the hypersurfaces $\xi_i = \text{const}$ represent characteristics, and the rows of matrix (1.4) are linearly dependent [14]. Let us determine the coefficients of this linear dependence, i.e. find the eigenvector of the transposed matrix, which corresponds to the zero eigenvalue. It can be defined by an expression of the form

$$\mathrm{S} = \{c^2 U_i;\ -c^2 A_{i2};\ -c^2 A_{i3};\ -c^2 A_{i4};\ U_i\} \qquad (1.5)$$

Multiplying Eqs. (1.3) by related projections of vector S and adding, we obtain

$$\sum_{j=1}^{4}\left[\rho c^2 (\mathbf{A}_j U_i - \mathbf{A}_i U_j) \frac{\partial \mathbf{V}}{\partial \xi_j} + (U_i U_j - c^2 \mathbf{A}_i \mathbf{A}_j) \frac{\partial p}{\partial \xi_j}\right] = \mathbf{L}\cdot \mathrm{S} \qquad (1.6)$$

Equation (1.6) contains only one derivative with respect ξ_i viz that of p, and, if ξ_i is a characteristic variable, there are no such derivatives. (For $j = i$ the coefficients at $\partial \mathbf{V} / \partial \xi_i$ are identically zero and the coefficient at $\partial p / \partial \xi_i$ is equal $\Delta_i / \rho^3 U_i$.)

We solve the system of Eqs. (1.3) by the method of small perturbations. We represent the dependent and independent variables in the following form:

$$\Omega_k = \Omega_k^\circ + \varepsilon^{\alpha_k} \Omega_k', \qquad \xi_j' = \varepsilon^{r_j} \xi_j \qquad (1.7)$$

where ε is some small parameter, Ω_k with $k = 1, \ldots, 5$ denotes, respectively, u, v, w, p and ρ, and α_k and r_j are constants with $j = 1, 2, 3, 4$. It is assumed that the following conditions:

$$\Omega_k^\circ = \text{const}, \ \alpha_k > 0, \ \rho^\circ, \ p^\circ \neq 0$$

are satisfied, and that the equality $\mathbf{V}^\circ = 0$ is only possible for nonstationary flows. The zero approximation Ω_k° can correspond either to a uniform oncoming stream (supersonic flow past a slender profile) or to some point of the flow field (e.g. values of parameters at the center of a nozzle).

Let us substitute (1.7) into (1.3) and consider separately each expression containing a partial derivative with respect to any variable, with only the term of lower order with respect to ε retained in it. As the result we obtain a system of linear equations in partial derivatives which superficially does not differ from (1.3). The difference is in that coefficients at derivatives $\partial\Omega_k' / \partial\xi_j'$ are replaced by their values calculated in the zero approximation and that each term is multiplied by ε in some power (we denote it by β_{jk}), which formally determines its order of smallness. From (1.3) and (1.7) we can obtain

$$\beta_{jk} = \alpha_k + r_j \tag{1.8}$$

It is convenient to consider all quantities in the derived equations as dimensionless. We relate the dimension of length to some characteristic linear dimension, velocity to c°, density to ρ°, pressure to $\rho^\circ c^{\circ 2}$, etc. We omit primes at perturbed quantities.

Let us assume that all terms in the derived equations are of the same order, which reduces (1.3) to the form

$$\begin{aligned} \mathbf{A}_i \frac{\partial \mathbf{V}}{\partial \xi_i} + U_i^\circ \frac{\partial \rho}{\partial \xi_i} &= L_1 - \sum_{j\neq i} \left(\mathbf{A}_j \frac{\partial \mathbf{V}}{\partial \xi_j} + U_j^\circ \frac{\partial \rho}{\partial \xi_j}\right) \\ U_i^\circ \frac{\partial \mathbf{V}}{\partial \xi_i} + \mathbf{A}_i \frac{\partial p}{\partial \xi_i} &= \mathbf{l} - \sum_{j\neq i} \left(U_j^\circ \frac{\partial \mathbf{V}}{\partial \xi_j} + \mathbf{A}_j \frac{\partial p}{\partial \xi_j}\right) \\ U_i^\circ \frac{\partial p}{\partial \xi_i} - U_i^\circ \frac{\partial \rho}{\partial \xi_i} &= L_5 - \sum_{j\neq i} U_j^\circ \left(\frac{\partial p}{\partial \xi_j} - \frac{\partial \rho}{\partial \xi_j}\right) \end{aligned} \tag{1.9}$$

It follows from condition $\beta_{jk} = \beta$ and (1.8) that all α_k and r_j are expressed in terms of two parameters: $\alpha_k = \alpha$ and $r_j = \beta - \alpha$. These conditions conform to the conventional linear theory (the flow is defined by a linear system of equations). Terms of first order with respect to ε ($\alpha = \beta = 1$) are considered in the classical linear theory with ε determined by boundary conditions.

Let us consider the matrix of coefficients at derivatives with respect to ξ_i appearing in (1.9) (it is formally the same as (1.4), if zero approximations are substituted for its elements). Condition $\Delta_i = 0$ is obviously determined by the choice of independent variables (quantities A_{ij}), as well as by the choice of the zero approximation (values of $\mathbf{V}^\circ$). Let us assume that in some region of the flow the following three conditions are satisfied:

$$1^\circ.\ U_i^\circ \neq 0, \quad 2^\circ.\ \Delta_i^\circ = 0, \quad 3^\circ.\ \partial\Omega_k/\partial\xi_i \gg \partial\Omega_k/\partial\xi_j \quad (i\neq j) \tag{1.10}$$

Conditions (1.10) are sufficient for making the conventional linear theory inapplicable in the indicated region. Condition 1° implies that at least one element of each row of matrix (1.4) is nonzero, condition 3° shows that derivatives with respect to ξ_j can be neglected in (1.9) as being considerably smaller than ξ_i. Finally, in virtue of 2° a linear dependence exists between the left-hand sides of Eqs. (1.9), hence the corresponding system is either incomplete or inconsistent (in the absence of such dependence between its right-hand sides).

2. Let us determine the flow regions which satisfy conditions (1.10) for a stationary plane flow of perfect gas ($\mathbf{L} = 0$). We consider the problem of flow past a slender profile, setting $\xi_2 = \xi$ and $\xi_3 = \eta$. We select the system of coordinates so that $v^\circ = 0$ and $u^\circ = M_\infty$. As the zero approximation we select parameter values of the steady oncoming stream. We then have $U_2^\circ = A_{22}M_\infty$, $U_3^\circ = A_{32}M_\infty$, $\omega_2^2 = A_{22}^2 + A_{23}^2$

and condition 2° becomes

$$A_{23}^2 = A_{22}^2 (M_\infty^2 - 1) \tag{2.1}$$

If ξ is the characteristic variable, then for a supersonic flow $(M_\infty > 1)$ condition (2.1) is satisfied. Conditions 1° and 2° are satisfied, for example, for $A_{22} = A_{33} = 1$, $A_{23} = -\sqrt{M_\infty^2 - 1}$ and $A_{32} = 0$, which is equivalent to the following selection of variables: $\eta = y$, $\xi = \xi_0 = x - y\sqrt{M_\infty^2 - 1}$. Thus the conventional linear theory is inapplicable in flow regions, where the derivatives with respect to ξ and η are of different order (condition 3°).

Let us formulate a modified linear theory for these regions [4]. Assuming that ξ is the characteristic variable of the input system (1.3) (i.e. $\Delta_2 = 0$ is satisfied exactly and not in the zero approximation), we find that derivatives with respect to ξ do not appear in (1.6). From (1.9) and (1.6) we obtain

$$u = -M_\infty^{-1} p = -M_\infty^{-1} \rho = (M_\infty^2 - 1)^{-1/2} v \tag{2.2}$$

$$M_\infty \frac{\partial v}{\partial \eta} + \sqrt{M_\infty^2 - 1}\,\frac{\partial p}{\partial \eta} = 0 \tag{2.3}$$

The solution for velocity is of the form $u = F(\xi)$ with function F determined by boundary conditions.

Setting $\xi = \xi_0 + \Psi(\xi_0, \eta)$ in accordance with the method of deformed coordinates [15], for the determination of Ψ from the condition $\Delta_2 = 0$ we obtain the equation

$$2\sqrt{M_\infty^2 - 1}\,\frac{\partial \Psi}{\partial \eta} + (\varkappa + 1) M_\infty^2 F(\xi_0) = 0$$

where $\varkappa$ is the adiabatic exponent of gas. The derived formulas determine the solution of the problem in the region in which the conventional linear theory is inapplicable.

Let us apply another method. We assume that $\xi = \xi_0$ is the characteristic variable of the linearized system of Eqs. (1.9), i.e. that $\Delta_2 = 0$ is satisfied only in the zero approximation. Neglecting smalls of higher order and allowing for (2.2) we obtain $\Delta_2 = (\varkappa + 1) M_\infty^3 u$. Using (2.2) and the formula for Δ_2, we reduce Eq. (1.6) to the nonlinear equation

$$(\varkappa + 1) M_\infty^3 u \frac{\partial u}{\partial \xi_0} + 2\sqrt{M_\infty^2 - 1}\,\frac{\partial u}{\partial \eta} = 0$$

The solution of this equation is exactly the same as that derived by the method of deformed coordinates. Thus in the case of a perfect gas the modified linear theory and the nonlinear theory of weakly perturbed flows yield the same results.

For transonic flows $(M_\infty = 1)$ formula (2.1) implies that $A_{23} = 0$. Let us assume for simplicity that $A_{32} = 0$, $A_{22} = A_{33} = 1$, i.e. $\xi = x$ and $\eta = y$, then $U_2^\circ = 1$, $U_3^\circ = 0$, $\omega_2^2 = 1$ and conditions 1° and °2 are satisfied. If we assume that condition 3° is also satisfied, then from the first two equations of (1.9) we obtain $p = \rho = -u$. To avoid a trivial solution it is necessary to set in the third equation of (1.9) $\partial v / \partial \xi = -\partial p / \partial \eta$. Carrying out on (1.6) the same transformations as in the previous case, we finally obtain

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y} = (\varkappa + 1) u \frac{\partial u}{\partial x}$$

This is the known equation of the theory of small perturbations in transonic flows.

The analysis of (1.8) and of the order of the nonlinear term yields $\alpha_1 = \alpha_3 = \alpha_4 = \alpha$, $\alpha_2 = 3\alpha/2$, $r_1 = \beta - \alpha$, $r_2 = \beta - \alpha/2$. It is usually assumed that $\alpha = \beta = 1$.

In that case $u = O(\varepsilon)$, $v = O(\varepsilon^{3/2})$, $x = O(1)$ and $y = O(\varepsilon^{1/2})$, which are known estimates of the theory of weakly perturbed transonic flows.

For subsonic flows $(M_\infty < 1)$ condition (2.1) cannot be satisfied, hence the considered regions of inapplicability of the linear theory do not exist in the case of subsonic speeds. The known problems of particular perturbations for equations of the elliptic kind [15] depend on boundary conditions.

3. Let us enumerate all regions of nonstationary space flows for which conditions (1.10) are satisfied, and derive the nonlinear equations which define weakly perturbed flows in these regions.

We assume that derivatives with respect to ξ_i are of the lowest order of smallness. Indices i and $^\circ$ will be omitted $(U_i^\circ = U,\ \omega_i^\circ = \omega,\ \xi_i = \xi$, etc.) whenever this does not lead to misunderstandings. We denote by β_j the lowest exponent of β_{jk} with $k = 1, 2, \ldots, 5\ (\beta_i = \beta)$. We consider A_{ij} as constants and assume that condition 2° is satisfied only in the zero approximation. Conditions 1°–3° can be written as

$$1^\circ.\ U \neq 0,\ 2^\circ.\ U^2 = \omega^2,\ 3^\circ.\ \beta < \beta_j \quad (j \neq i) \tag{3.1}$$

Five fundamentally different cases exist when conditions (3.1) are satisfied.

First case. $\beta_{ik} = \beta$ for any k, i.e. all derivatives with respect to ξ are of the same order. From (1.8) we directly obtain that $\alpha_k = \alpha$, $\beta_{jk} = \beta_j$, $r = \beta - \alpha$ and $r_j = \beta_j - \alpha$. After the substitution of (1.7) into system (1.3), using the derived equalities and neglecting right-hand parts (see below), we obtain the following solution:

$$\rho = p,\quad U\mathbf{V} = -\mathbf{A}_i p \tag{3.2}$$

From these formulas and condition 2° we can obtain

$$\Delta_i = -U\omega^2\gamma p \tag{3.3}$$

where γ is a constant (for a perfect gas $\gamma = \varkappa + 1$).

When applying the method of small perturbations to (1.6), we assume that the nonlinear term containing a derivative with respect to ξ is of the same order as derivatives with respect to ξ_j, hence $\alpha + \beta = \beta_j$. Thus the order of magnitude of parameters is in this case

$$\alpha_k = \alpha,\ r = \beta - \alpha,\ r_j = \beta \quad (j \neq i) \tag{3.4}$$

Equation (1.6) assumes the form

$$-2\sum_{j\neq i} B_j \frac{\partial p}{q\xi_j} - \omega^2\gamma p \frac{\partial p}{\partial \xi} = \mathbf{L}\cdot\mathrm{S},\quad B_j = \mathbf{A}_i\mathbf{A}_j - U_iU_j \tag{3.5}$$

Note that, if the equality $\alpha + \beta = \beta_j$ is satisfied for a particular j there are no derivatives with respect to other variables under the summation sign in (3.5).

It follows from (3.4) and (3.5) that $\mathbf{L} = O(\varepsilon^{\alpha+\beta})$. Because of this the right-hand parts of (1.3) are not taken into account in the derivation of (3.2).

Second case. $\beta_{ik} = \beta\ (k \neq n)$ and $B_j \neq 0$ for at least one value of j. In such a case all derivatives with respect to ξ, except $\partial\Omega_n / \partial\xi$, are of the same order. It follows from (1.4) that system (1.3) has a trivial solution for $n = 4, 5$ (all derivatives with respect to ξ are zero), hence we assume $n = 1, 2, 3$. Below we consider, for simplicity, the case of $n = 2$.

The reason for using condition $B_j \neq 0$ is to make it possible to equate the order of the nonlinear term to that of the derivative $\partial p / \partial \xi_j$. To avoid a trivial solution it is necessary to set in the third of Eqs. (1.3) $A_{i3} = 0$. The final form of the system of equations is

$$\rho = p = -\frac{U}{A_{i2}} u = -\frac{U}{A_{i4}} w, \quad U \frac{\partial v}{\partial \xi} = -\sum_{j \neq i} A_{j3} \frac{\partial p}{\partial \xi_j} \tag{3.6}$$

$$A_{i3} = 0, \quad -2 \sum_{j \neq i} B_j \frac{\partial p}{\partial \xi_j} - \omega^2 \gamma p \frac{\partial p}{\partial \xi} = \mathrm{L \cdot S}$$

The order of terms is

$$\alpha_k = \alpha, \ \alpha_n = 2\alpha, \ r = \beta - \alpha, \ r_l = \beta, \ r_j \geqslant \beta \quad (j \neq i, l) \tag{3.7}$$

If $B_l = 0 \ (l \neq i, j)$, the corresponding terms simply vanish from the last of Eqs. (3.6).

Third case. $\beta_{ik} = \beta \ (k \neq n)$ and $B_l = 0$ for at least one value of l. For $B_l = 0$ it is possible to equate the order of the nonlinear term to that of highest derivative with respect to ξ_l. For $n = 2$ $\partial v / \partial \xi_l$ is such a derivative. Thus $\beta + \alpha = \alpha_n + r_l$. Furthermore, the condition of nontriviality of solution of the third of Eqs. (1.3) implies that $\alpha_n + \beta - \alpha = \alpha + r_l$. Taking into account (1.8), we finally obtain

$$\alpha_k = \alpha, \ \alpha_n = 3\alpha / 2, \ r = \beta - \alpha, \ r_l = \beta - \alpha / 2, \ r_j \geqslant \beta \ \text{for} \tag{3.8}$$

$$B_j \neq 0, \ r_j \geqslant \beta - \alpha / 2 \ \text{ for } \ B_j = 0 \quad (j \neq i, l)$$

The fundamental equations that define the flow in this region are

$$\rho = p = -\frac{U}{A_{i2}} u = -\frac{U}{A_{i4}} w, \quad A_{i3} = 0, \quad U \frac{\partial v}{\partial \xi} = -\sum_{l \neq i} A_{l3} \frac{\partial p}{\partial \xi_l} \tag{3.9}$$

$$-2 \sum_{j \neq i, l} B_j \frac{\partial p}{\partial \xi_j} + U \sum_{l \neq i} A_{l3} \frac{\partial v}{\partial \xi_l} - \omega^2 \gamma p \frac{\partial p}{\partial \xi} = \mathrm{L \cdot S}$$

Subscripts l and j relate to variables ξ_l for which $B_l = 0$, and to those for which $B_j \neq 0$, respectively. Note that all coefficients B can vanish only in the case of a stationary transonic flow. In the case of a nonstationary flow this condition would imply that the Jacobian of transformation (1.2) is zero.

Fourth case. $\beta_{ik} = \beta \ (k \neq n, s)$ and $B_j \neq 0$ for at least one value of j. All derivatives with respect to ξ, except two, are of the same order. A nontrivial solution exists only for $n, s = 1, 2, 3$. Setting $n = 2$ and $s = 3$, we find from the third and fourth of Eqs. (1.3) that $A_{i3} = A_{i4} = 0$. Applying the same reasoning as in the previous case, we obtain the system of equations

$$\rho = p = -\frac{U}{A_{i2}} u, \quad U \frac{\partial v}{\partial \xi} = -\sum_{j \neq i} A_{j3} \frac{\partial p}{\partial \xi_j}, \quad U \frac{\partial w}{\partial \xi} = -\sum_{j \neq i} A_{j4} \frac{\partial p}{\partial \xi_j} \tag{3.10}$$

$$A_{i3} = A_{i4} = 0, \quad -2 \sum_{j \neq i} B_j \frac{\partial p}{\partial \xi_j} - \omega^2 \gamma p \frac{\partial p}{\partial \xi} = \mathrm{L \cdot S}$$

If $B_l = 0 \ (l \neq i, j)$, the related term simply vanishes in the last of Eqs. (3.10). In this case the order of terms is as follows:

$$\alpha_k = \alpha \ (k \neq n, s), \ \alpha_n = \alpha_s = 2\alpha, \ r = \beta - \alpha, \ r_j \geqslant r_l = \beta \tag{3.11}$$

It should be noted that at first glance L_n and L_s should have been retained in the right-hand parts of the third and fourth of Eqs. (3.10). However, a closer examination of relaxation processes shows that they can be neglected.

Fifth case. $\beta_{ik} = \beta\ (k \neq n, s)$ and $B_l = 0$. We ascribe subscript l to those variables for which $B_l = 0$ and subscript j to those for which $B_j \neq 0$. Assuming as in the previous case that the order of the nonlinear term is the same as that of the term containing $\partial v / \partial \xi_l$ or $\partial w / \partial \xi_l$, (with $n = 2$ and $s = 3$) we obtain

$$\rho = p = -\frac{U}{A_{i2}} u, \quad U\frac{\partial v}{\partial \xi} = -\sum_{l \neq i} A_{l3} \frac{\partial p}{\partial \xi_l}, \quad U\frac{\partial w}{\partial \xi} = -\sum_{l \neq i} A_{l4} \frac{\partial p}{\partial \xi_l} \quad (3.12)$$

$$A_{i3} = A_{i4} = 0, \quad -2\sum_{j \neq i} B_j \frac{\partial p}{\partial \xi_j} + U \sum_{l \neq i} \frac{\partial}{\partial \xi_l}(A_{l3} v + A_{l4} w) - \omega^2 \gamma p \frac{\partial p}{\partial \xi} = \mathbf{L} \cdot \mathbf{S}$$

In this case all remarks about B made in the analysis of the third case are valid. The order of parameters is as follows:

$$\alpha_k = \alpha, \ \alpha_n = \alpha_s = 3\alpha / 2, \ r = \beta - \alpha, \ r_l = \beta - \alpha / 2, \ r_j = \beta \quad (3.13)$$

Further increase of the number of derivatives of higher orfer with respect to ξ yields trivial equations.

It follows from the above calculations that in all cases

$$\mathbf{L} \cdot \mathbf{S} = UL_1^\circ - \mathbf{A}_i \mathbf{l}^\circ + UL_5^\circ = O(\varepsilon^{\alpha+\beta}) \quad (3.14)$$

4. Let us consider a homogeneous or heterogeneous medium in which N unsteady processes can take place (oscillatory relaxation, chemical reactions, mismatch of velocities and temperatures of solid or liquid particles and gas, interphase mass transfer, etc.).

We denote the relaxation parameter (completeness of an unsteady process) by q_j and the affinity of such process by $Q_j = q_{je} - q_j$ (q_{je} is the equilibrium value of the relaxation parameter). We omit the general expression for $\mathbf{L}$, and present it directly in the linearized form. For this we segregate all relaxation processes into several kinds. We assume that for small variation of flow parameters k relaxation processes weakly deviate from the equilibrium state (near-equilibrium flow $Q_k^\circ = 0$), $l + m$ processes proceed very slowly (near-frozen state, $Q_l^\circ = 0$, $Q_m^\circ \neq 0$), and n relaxation processes may pass from the frozen to the equilibrium state ($Q_n^\circ = 0$). The expression for $\mathbf{L} \cdot \mathbf{S}$ can be presented in the following general form:

$$\mathbf{L} \cdot \mathbf{S} = U^2 \sum_k H_k \frac{\partial Q_k}{\partial \xi} + U^2 \sum_{j=l,m,n} H_j \frac{\partial q_j}{\partial \xi} \quad (k + l + m + n = N) \quad (4.1)$$

Application of the method of small perturbations to kinetic equations yields [9, 11]

$$U\frac{\partial q_j}{\partial \xi} = Q_j^\circ \Lambda_j^{-1} + Q_j \Lambda_j^{-1} \quad (4.2)$$

where Λ_j are relaxation times.

In what follows equilibrium values of relaxation parameters will be presented in the form

$$q_{je} = E_j p \quad (4.3)$$

Equations (4.1) and (4.2) will be used in flow regions in which the conventional linear theory is inapplicable (i.e. in regions corresponding to the five considered cases, hence for nonequilibrium processes dependent on energy exchange Eq. (4.3) follows from $q_e = q_e(\rho, p)$, since for all five cases $\rho = p$. If the nonequilibrium processes depend on the exchange of momentum, then $q_e = u$, v, or w [16], and the velocity components are either expressed in terms of p, or do not appear in $L \cdot S$ (in consequence of $A_{i3} = 0$ or $A_{i4} = 0$).

With the use of (4.3) the kinetic equations (4.2) can be presented in the following two alternative forms:

$$\frac{\partial Q_k}{\partial \xi} = E_k \frac{\partial p}{\partial \xi} - \frac{Q_k}{U\Lambda_k}, \quad U\Lambda_j \frac{\partial q_j}{\partial \xi} = Q_j^\circ + E_j p - q_j \quad (j = l, m, n) \qquad (4.4)$$

Knowing the order of parameters $\xi = O(\varepsilon^{\beta-\alpha})$, $p = O(\varepsilon^{\alpha})$ and $L \cdot S = O(\varepsilon^{\alpha+\beta})$, which are the same for all five of the considered regions of linear theory inapplicability, we can estimate the order of parameters which define relaxation processes. Let us assume that $H = O(\varepsilon^s)$, then from (4.1) we directly obtain $Q_k = O(\varepsilon^{2\alpha-s})$, and $q_j = O(\varepsilon^{2\alpha-s})$ $(j = l, m, n)$.

Assuming that $Q_m^\circ = O(1)$ and retaining only terms of lower order, we obtain

$$\begin{aligned} &Q_k = \Lambda_k E_k U \frac{\partial p}{\partial \xi}, \quad q_k = E_k p, \quad \Lambda_k = O(\varepsilon^{2\alpha-\beta-s}) \\ &U \frac{\partial q_l}{\partial \xi} = E_l \Lambda_l^{-1} p, \quad Q_l = E_l p, \quad \Lambda_l = O(\varepsilon^{s-\beta}) \\ &\frac{\partial q_m}{\partial \xi} = \overset{\circ}{Q}_m \Lambda_m^{-1}, \qquad \Lambda_m = O(\varepsilon^{s-\beta-\alpha}) \\ &U\Lambda_n \frac{\partial q_n}{\partial \xi} = E_n p - q_n, \quad s = \alpha, \quad \Lambda_n = O(\varepsilon^{\alpha-\beta}) \end{aligned} \qquad (4.5)$$

For generality, estimates for Λ_j are shown in (4.5), however, if $\Lambda_j = O(1)$, then $s = 2\alpha - \beta$, $s = \beta$, $s = \beta + \alpha$, $s = \alpha = \beta$, respectively, for each of the four kinds of relaxation processes. We introduce the notation

$$\mu = U^2 \sum_k \Lambda_k E_k H_k, \quad \chi = -\sum_l \Lambda_l^{-1} E_l H_l, \quad \delta = -\sum_m \Lambda_m^{-1} E_m H_m$$

and substitute the first three formulas of (4.5) into (4.1), then

$$U^{-1} L \cdot S - \mu \frac{\partial^2 p}{\partial \xi^2} + \chi p + \delta = U \sum_n H_n \frac{\partial q_n}{\partial \xi} = K \qquad (4.6)$$

Using formula $q_n = E_n p + Q_n$ and the first of Eqs. (4.4), we eliminate q_n from (4.6). As the result we obtain

$$(-1)^n a_0 K + \sum_{r=1}^{n} \frac{\partial^r}{\partial \xi^r} [(-1)^{n-r} a_r K + b_r p] = 0 \qquad (4.7)$$

where a_r are coefficients of the polynomial $P(x) = (x - U^{-1}\Lambda_1^{-1}) \ldots (x - U^{-1}\Lambda_n^{-1})$, and b_r is obtained in the course of derivation of Eq. (4.7).

Note that in constructing the nonlinear theory of weakly perturbed flows in conformity with formula (4.1) for $L \cdot S$ it is necessary to use the equilibrium speed of sound for relaxation processes, and the frozen for all others. The condition for absence of derivatives of Ω_j in L is satisfied in this case.

5. Using (4.6), (4.7), (3.5), (3.6), (3.9), (3.10) or (3.12) it is possible to derive various nonlinear equations which define the flow in regions where the conventional linear theory is inapplicable.

Let us consider, for instance, a nonstationary supersonic space flow with relaxation processes of the first three kinds. Conditions (1.10) are satisfied for $A_{11} = A_{22} = A_{33} = A_{44} = 1,\ A_{21} = -2M_\infty,\ A_{24} = -B_\infty\ (u^\circ = M_\infty,\ v^\circ = w^\circ = 0,\ B_\infty{}^2 = M_\infty{}^2 - 1)$, and consequently $U_1 = 1,\ U_2 = -M_\infty$ and $U_3 = U_4 = 0$. This corresponds to variables $\xi_1 = t,\ \xi_2 = \xi = -2M_\infty t + x - B_\infty z,\ \xi_3 = y$ and $\xi_4 = z$. Equations

$$\frac{\partial v}{\partial \xi} = \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial t} - \frac{B_\infty}{M_\infty}\frac{\partial u}{\partial z} + \frac{\gamma}{2} M_\infty^2 u \frac{\partial u}{\partial \xi} = \frac{\mu}{2}\frac{\partial^2 u}{\partial \xi^2} - \frac{\chi}{2} u + \frac{\delta}{2M_\infty}$$

define the flow in the region in which in the second case the conventional linear theory is inapplicable. The order of magnitude of parameters is determined by formulas (3.7).

Let us consider the nonstationary transonic space flow. Conditions (1.10) are satisfied, if $A_{11} = A_{22} = A_{33} = A_{44} = M_\infty = 1$, and the remaining A_{ij} are set equal to zero. We then have $U_1 = U_2 = 1,\ U_3 = U_4 = 0,\ B_1 = -1$ and $B_2 = B_3 = B_4 = 0$. For relaxation processes close to either the frozen or equilibrium state from (3.12) and (4.6) we obtain

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial y}, \quad -2\frac{\partial u}{\partial t} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \gamma u \frac{\partial u}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} - \chi u + \delta = 0$$

These equations were analyzed by Ryzhov [17] for a perfect gas.

Note that in this case it is not necessary to assume in the zero approximation $M_\infty = 1$. Thus, for example, it is possible to assume $U^2 - \omega^2 = O(\varepsilon^\alpha)$ and, using the velocity potential φ, for a single relaxation process of the fourth kind from (3.12) with allowance for (4.7) to obtain

$$\Lambda \frac{\partial}{\partial x}\left[\left(M_\infty^2 - 1 - \gamma \frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - 2\frac{\partial^2 \varphi}{\partial t \partial x}\right] +$$
$$\left(M_e^2 - 1 - \gamma_e \frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - 2\frac{\partial^2 \varphi}{\partial t \partial x} = 0$$

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METHOD OF DERIVATION OF THE KORTEWEG – de VRIES – BURGERS EQUATION

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A method of derivation of the Korteweg – de Vries – Burgers (KdVB) equation for media with dispersion and dissociation, whose behavior is defined by equations of a fairly general form, is presented. The method is used for obtaining KdVB equations for collision plasma with Hall dispersion and the Korteweg – de Vries (KdV) equation for waves propagating in hot collisionfree plasma across a magnetic field.

Considerable attention was recently devoted to the investigation of the Korteweg – de Vries equation which provides a good definition of weakly nonlinear waves in the presence of dispersion in various media waves on shallow water, ionization sound in plasma, etc.). Since this equation is at present well known, its derivation is important for the investigation of wave motion in any medium. It was stated [1] on the basis of investigation of a number of examples that the